

A Straightforward Explanation of the Mathematical Foundation of the Analytic Hierarchy Process (AHP)

This is a full methodological briefing with all of the math
and background from the founders of AHP at Decision Lens Inc.

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Decision Lens

Better Decisions for Better Performance

Introduction

- This presentation assumes that the reader has a basic level of understanding of the Analytic Hierarchy Process (AHP).
- The AHP is the most widely used decision theory for establishing priorities among a set of elements¹.
- Decision Lens is the software used to enable AHP-based decision-making.

<http://www.decisionlens.com>

AHP Overview

- Key characteristics of AHP's calculations:
 - Decision-makers trade-off the importance of elements, one to another, in a pairwise fashion.
 - The judgments are entered into a pairwise comparison matrix (table of rows and columns).
 - Each position in the matrix contains a number that is a ratio of how important the row element is to the column element.
 - The calculations (described hereafter) are used to derive the priorities for all of the elements.
 - Some inconsistency is inevitable in making comparisons, and is allowed to some degree by AHP (generally less than 10% inconsistency).

Agenda

- Part I: Slides 5 through 23 provide background on matrices and theory
 - what they are
 - how to operate with them (i.e. multiply together)
 - this background is useful for Part II where we will show how the AHP uses the matrix.
- Part II: Slides 24 onwards focuses specifically on how AHP and Decision Lens calculate priorities and inconsistency.


Part I: We Start with an Introduction to Matrices – A Matrix is Used to Solve a System of Equations

- We want to determine the values of X and Y
- 2 simple equations to solve for X and Y are:

$$2X + 3Y = 6$$

$$X + 4Y = 7$$

These equations can be represented in a matrix as follows:


$$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \xrightarrow{\text{Multiplied Times}} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \end{bmatrix}$$

This is known as an “inhomogeneous system” of linear equations, meaning that there are some constant terms on the right side (they aren’t variables, they are actual numbers and aren’t all 0)

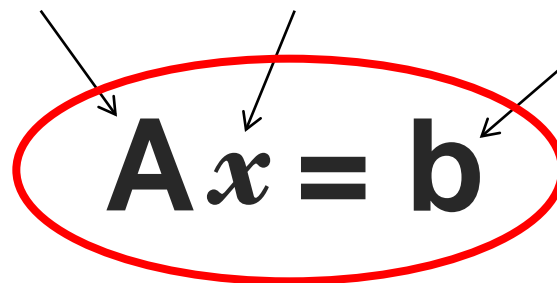
If we put this calculation into a simple equation, it reads as...

$$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \xrightarrow{\text{Multiplied Times } \rightarrow} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \end{bmatrix}$$

This matrix
can be represented
by this letter...

The variables by
this letter...

And the constants by
this letter...


$$\mathbf{A}x = \mathbf{b}$$

We know from algebra that if...

$$2x = 3 \text{ then } x = \frac{1}{2} 3$$

Remember that the $\frac{1}{2}$ can also be represented as 2^{-1}

We can also show this equation as

$$x = 2^{-1} 3$$

This is called the “inverse” of 2

Going back to our equation

Instead of this we will
just use x

$$\begin{bmatrix} x \\ y \end{bmatrix}$$

$$Ax = b$$

We can now write it as

$$x = A^{-1}b$$

This is called the “inverse” of “A”
which is the **Inverse of *matrix A***

This is important – we will use it in a minute

The matrix times it's inverse gives us the “Identity” matrix, which also can be thought of as a generalization of “1”

$$\mathbf{A} \xrightarrow{\text{Multiplied Times}} \mathbf{A}^{-1} = \mathbf{I}$$

↖ This is called the “Identity” of the matrix and is just a fancy way of saying that the matrix Multiplied by it's inverse gives you a “1”

Because the Identity is “1”, anytime you multiply the matrix by its Identity you get the matrix itself back...

$$\mathbf{AI} = \mathbf{IA} = \mathbf{A}$$

How do we get the inverse of the matrix?

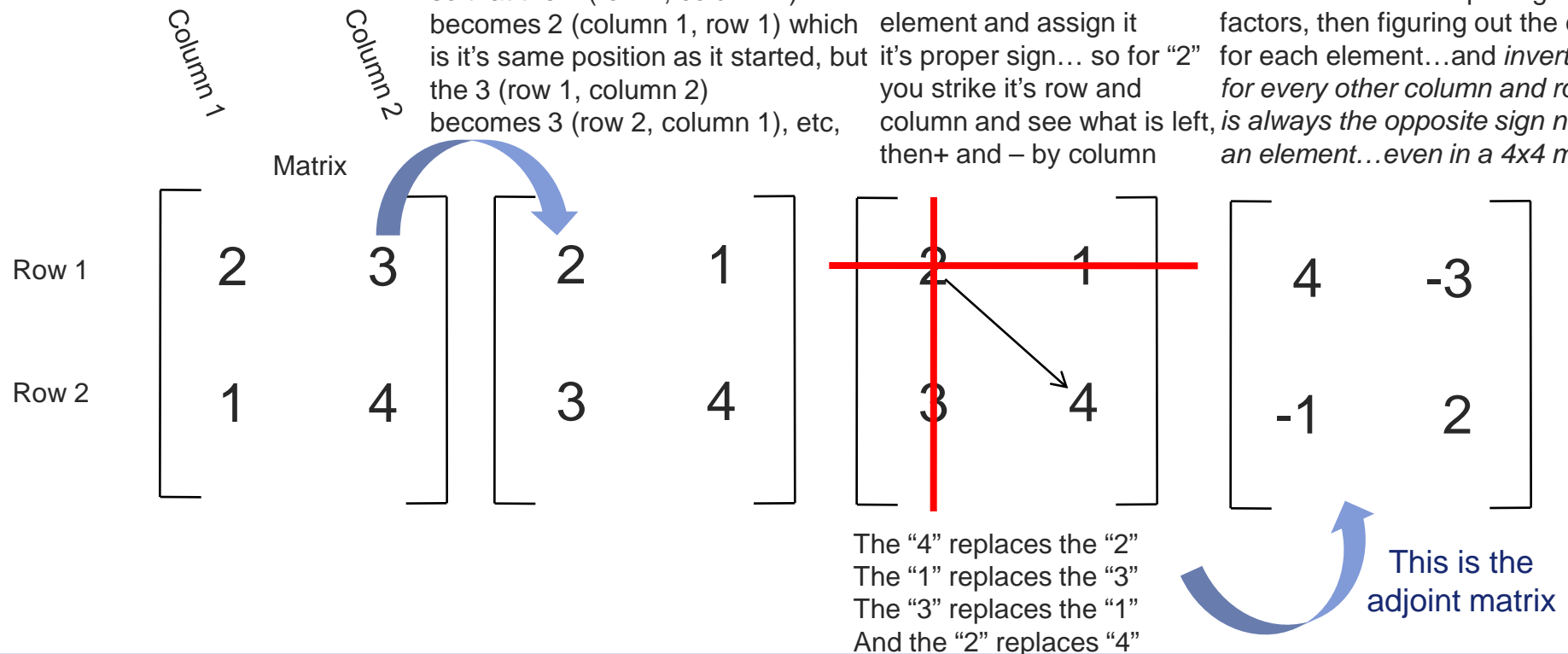
Step 1: Establish the “Adjoint” of the Matrix

So think of it this way with our original example...in the inverse what were rows become columns, and what were columns become rows...this is a systematic way of doing elimination in simultaneous equations...

First you transpose it so that the 2 (row 1, column 1) becomes 2 (column 1, row 1) which is it's same position as it started, but the 3 (row 1, column 2) becomes 3 (row 2, column 1), etc,

Now you have to find the determinant for each element and assign it it's proper sign... so for “2” you strike it's row and column and see what is left, then + and - by column

Now you have the “adjoint” of the matrix which is transposing the factors, then figuring out the determinant for each element...and *inverting the sign for every other column and row so is always the opposite sign next to an element...even in a 4x4 matrix,*



How do we get to the inverse of the matrix?

Step 2: Find the Determinant of Your Original Matrix

The determinant helps you calculate the inverse of the matrix by creating a common number to divide all of the numbers in the adjoint matrix by → this is placed in the “denominator”

Matrix

$$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \quad \text{minus} \quad \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$$

The diagram illustrates the calculation of the determinant for a 2x2 matrix. The first matrix is $\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$. An arrow labeled "Multiplied Times" points from the element 2 to the element 4. The second matrix is $\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$. An arrow labeled "Multiplied Times" points from the element 3 to the element 1. The word "minus" is placed between the two matrices.

$$(2 \times 4) - (3 \times 1) = 5$$

→ This is called the “determinant” of the matrix and can be used as a common denominator to help figure out the matrix’ inverse

How do we get the inverse of the matrix?

Step 3: From the Adjoint and Determinant, we Build the Inverse Matrix

Adjoint

$$\begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$$

Divided by Determinant

$$\begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix} \div 5 =$$

This is the inverse matrix

$$\begin{bmatrix} \frac{4}{5} & \frac{-3}{5} \\ \frac{-1}{5} & \frac{2}{5} \end{bmatrix}$$

Remember the determinant? It was a "5". This Becomes the denominator in the inverse matrix.

Now we can show how to get the Identity Matrix by Multiplying

$$\mathbf{A} \xrightarrow{\text{Multiplied Times}} \mathbf{A}^{-1} = \mathbf{I}$$

This is called the “Identity” matrix and Is just a fancy way of saying that the matrix Multiplied by it’s inverse gives you a “1”

Column 1 Column 2

Matrix

Row 1

Row 2

multiplied times

Inverse Matrix

Identity Matrix

$$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} \frac{4}{5} & \frac{-3}{5} \\ \frac{-1}{5} & \frac{2}{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

But how did we get these numbers? You have to know how to multiply matrices....

How to multiply one matrix by another so that we know we have the inverse matrix correct...

Take Row 1...times...Column 1 =

The diagram illustrates the calculation of the first element of the identity matrix. It shows three matrices: the original matrix, the inverse matrix, and the resulting identity matrix. The original matrix is $\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ with the first row circled. The inverse matrix is $\begin{bmatrix} \frac{4}{5} & -\frac{3}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{bmatrix}$ with the first column circled. The text "multiplied times" is between them. The identity matrix is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ with an arrow pointing from the top-left element (1) to the calculation below. The calculation is $(2 \times \frac{4}{5}) + (3 \times -\frac{1}{5}) = \frac{8}{5} + (-\frac{3}{5}) = \frac{5}{5} = 1$.

$$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \text{ multiplied times } \begin{bmatrix} \frac{4}{5} & -\frac{3}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$(2 \times \frac{4}{5}) + (3 \times -\frac{1}{5}) = \frac{8}{5} + (-\frac{3}{5}) = \frac{5}{5} = 1$$

Take Row 1...times...Column 2 =

The diagram illustrates the calculation of the second element of the identity matrix. It shows the same three matrices as the previous step. The original matrix is $\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ with the first row circled. The inverse matrix is $\begin{bmatrix} \frac{4}{5} & -\frac{3}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{bmatrix}$ with the second column circled. The text "multiplied times" is between them. The identity matrix is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ with an arrow pointing from the top-right element (0) to the calculation below. The calculation is $(2 \times (-\frac{3}{5})) + (3 \times \frac{2}{5}) = -\frac{6}{5} + (\frac{6}{5}) = 0$. A text box on the right says: "Then take row 2 x column 1 and row 2 x column 2 for the row 2 positions and you have calculated the Identity matrix..."

$$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \text{ multiplied times } \begin{bmatrix} \frac{4}{5} & -\frac{3}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$(2 \times (-\frac{3}{5})) + (3 \times \frac{2}{5}) = -\frac{6}{5} + (\frac{6}{5}) = 0$$

Then take row 2 x column 1 and row 2 x column 2 for the row 2 positions and you have calculated the Identity matrix...

Now that we have the inverse matrix, we can use it to figure out what X and Y are...

Remember that this
"x" stands for the
Variables x and y

...and A^{-1} is
the inverse of the matrix A

...and the b is
the constant

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{4}{5} & -\frac{3}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{bmatrix} \begin{bmatrix} 6 \\ 7 \end{bmatrix}$$

Use the same multiplication process to
get to X and Y

$$X = ((4/5) \times 6) + ((-3/5) \times 7) = 3/5 \text{ so } X = .6$$

$$Y = ((-1/5) \times 6) + ((2/5) \times 7) = 8/5 \text{ so } Y = 1.6$$

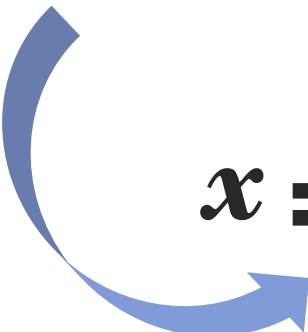
Therefore, to get the solution of an inhomogeneous system...

We must be able to calculate the inverse of “A” and thus the determinant of A must not be 0.

To obtain a non-zero solution of a homogeneous system of linear equations (that means that there are no constant terms) **the inverse matrix must not exist.**

Why?

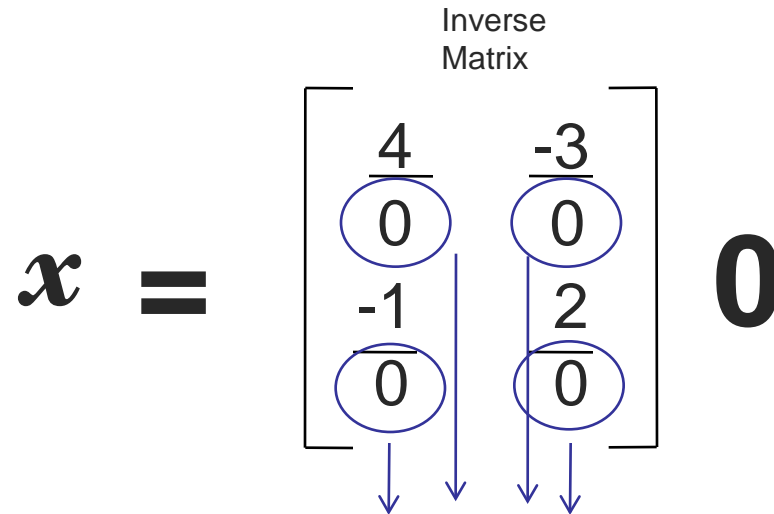
Remember that A^{-1} is the inverse matrix

$$\begin{aligned} \mathbf{x} &= \mathbf{A}^{-1} \mathbf{0} \\ \mathbf{x} &= \begin{bmatrix} x \\ y \end{bmatrix} = \begin{matrix} \text{Inverse} \\ \text{Matrix} \end{matrix} \begin{bmatrix} \frac{4}{5} & \frac{-3}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$


So when the inverse exists, we always get a zero solution!

We have to figure out for a way for this inverse matrix **NOT** to exist because if it does and it multiplies by 0 then x is also will be 0 and we have a solution of 0. That's bad.

If the determinant is a 0, then it looks like this...

$$x = \begin{matrix} & \text{Inverse} \\ & \text{Matrix} \\ \begin{bmatrix} \frac{4}{\textcircled{0}} & \frac{-3}{\textcircled{0}} \\ -1 & 2 \\ \textcircled{0} & \textcircled{0} \end{bmatrix} & \mathbf{0} \end{matrix}$$


Goes to a non-number and so the inverse matrix cannot be solved...
and that means that *x has a real value and is not just “0”*

So here is where we are. We don't want an inverse matrix. So we need a determinant of 0. How do we find a determinant of 0?

$$x = \cancel{A^{-1}} 0$$

...we don't
want this to
exist.

The way to find out a determinant (that is the denominator in the inverse matrix) where it is 0 is to find something called the *Eigenvalue* of the matrix.

How does an Eigenvalue problem arise?

- Pairwise comparisons make matrix

$$\begin{bmatrix} 6/6 & 6/3 & 6/2 \\ 3/6 & 3/3 & 3/2 \\ 2/6 & 2/3 & 2/2 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 6/6 & 6/3 & 6/2 \\ 3/6 & 3/3 & 3/2 \\ 2/6 & 2/3 & 2/2 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix} = 3 \begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix}$$

So when you multiply it out you get 3 6's from the first row, so it is 3 x 6 in the top position; second row is 3 3's so you have 3 x 3; third row is 3 2's so you have 3 x 2. So you factor the 3 out.

$$Ax = 3x$$

$$Ax = 3x$$

$$Ax = 3Ix$$

$$Ax = 3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x \quad Ax = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} x$$

A matrix times X equals another matrix times X,
So if we subtract them

$$(A-3I)x = 0$$

This is a system of homogeneous equations. It has a **non-zero solution** if it's determinant is zero.

The determinant of the matrix $A - 3I$ that is of the matrix is a cubic polynomial of the number 3, and we want it to be equal to zero (for the denominator in the inverse matrix) so we get a non-zero solution.

$$\begin{bmatrix} 6/6-3 & 6/3 & 6/2 \\ 3/6 & 3/3-3 & 3/2 \\ 2/6 & 2/3 & 2/2 - 3 \end{bmatrix}$$

This polynomial is known as the “characteristic” equation of the matrix A , and it's roots of which there are n (in this case three), are known as the ***Eigenvalues*** of A . In this case, 3 is actually an Eigenvalue. As it turns out, the Eigenvalue of a consistent matrix is always equal to it's dimension (number of rows)

Perron said that a positive matrix has a largest real positive eigenvalue (called the principal eigenvalue, denoted by λ_{\max}) that dominates all other eigenvalues in their absolute value.

$$\begin{bmatrix} 6/6-3 & 6/3 & 6/2 \\ 3/6 & 3/3-3 & 3/2 \\ 2/6 & 2/3 & 2/2-3 \end{bmatrix}$$

λ_{\max}

And, corresponding to this principal eigenvalue is a principal eigenvector that is positive and is obtained as a limiting power of the matrix. λ_{\max} is the largest root that will solve the characteristic equation.

Part II: AHP Calculations

There is something called the “Eigenvector” -- Eigen means “Self” in German so this is the “Self”vector -- which is the **priorities of the variables**. In a **consistent** matrix, it is easy to find the Eigenvector by just adding across the rows and normalizing.

Consistent Matrix		Step 1: Add across each row	Step 2: Then add down	Step 3: Then normalize	
1	3	$1 + 3 = 4$	$= 12/3$	$(12/3)$	$= .75$
$1/3$	1	$1/3 + 1 = 1 \frac{1}{3}$	$= 4/3$	$(16/3)$	
			$= 16/3$	$(4/3)$	$= .25$
				$(16/3)$	

This is the Eigenvector of this Matrix:

$$X = .75 \text{ and } Y = .25$$

It is difficult to be perfectly consistent, especially when trading off “intangible” variables such as...

- To pick a fruit, you might make trade-off comparisons between:



sweetness

tartness

texture

juiciness

size

- As you compare each of these elements to the others and put those ratios into a matrix, is it possible that you would be perfectly consistent in your comparison?
- Yes, it is possible but **not very likely**.

So if the Eigenvector for a Consistent Matrix Gives You the Priorities of the Elements, Can We Use It for an Inconsistent Matrix?

- Yes, in a matrix that is inconsistent, we also derive the priorities through the Eigenvector (we'll explained how we calculate that with inconsistency in a minute)
- But for an inconsistent matrix you want to know ***both***:
 - The priorities from the Eigenvector.
 - A measure of how ***consistent*** those priorities are.

This measure is called the “***consistency index***”

But in Decision-Making we often have inconsistent judgments...

- If the matrix has many variables, say 4, 5, 6, or 7, you have to make more and more judgments.
- It is likely that there will be inconsistency in how you have prioritized elements over one another.
 - (another way of looking at this is that the rows in the matrix will not be exact multiples of one another).
 - Does the following matrix look perfectly consistent?

	I	II	III	IV
I	1	1/7	1/5	1/3
II	7	1	5	5
III	5	1/5	1	5
IV	3	1/5	1/5	1

Element II is 5 times Element III and also 5 Times Element IV

BUT

Element III is judged As 5 times more important Than element IV when put Head to head....**inconsistent.**

But in Decision-Making we often have inconsistent judgments...

- Perron proved that if a matrix is positive, then its **largest eigenvector, the values of the variables**, is obtained by raising it to powers (multiplying it by itself).
- Each time you raise the matrix to a power, you find that the priorities, shown as the Eigenvector, that we calculated in the previous slide changes.
- But as you continue to multiply it out, the Eigenvector of each successive matrix changes less and less until it converges on one stable Eigenvector, or vector of priorities.

Calculations showing Eigenvector Converging when Inconsistent Matrix is Raised to Powers (Decision Lens Calcs to 32nd Power)

Matrix of Comparisons					Same Matrix					Squared Matrix Multiplied together					Fourth Power Multiplied together						
I	II	III	IV		Total	Normalized Priorities	I	II	III	IV		Total	Normalized Priorities		Total	Normalized Priorities					
I	1	1/7	1/5	1/3	1.676190	0.047516	I	1.000000	0.142857	0.200000	0.333333	4.000000	0.392381	1.180952	2.380952	7.954286	0.045160				
II	7	1	5	5	18.000000	0.510259	II	7.000000	1.000000	5.000000	5.000000	54.000000	4.000000	12.400000	37.333333	107.733333	0.611645				
III	5	1/5	1	5	11.200000	0.317495	III	5.000000	0.200000	1.000000	5.000000	26.400000	2.114286	4.000000	12.666667	45.180952	0.256510				
IV	3	1/5	1/5	1	4.400000	0.124730	IV	3.000000	0.200000	0.200000	1.000000	8.400000	0.868571	2.000000	4.000000	15.268571	0.086686				
					35.276190																

To Calculate the Inconsistency of the Matrix, You Have to Find the Largest Root (λ_{\max}) that Makes the Equations 0

	I	II	III	IV
I	1	1/7	1/5	1/3
II	7	1	5	5
III	5	1/5	1	5
IV	3	1/5	1/5	1

Step 1:
Add down each column

Column Sum 16.000000 1.542857 6.400000 11.333333

Eigenvector from
raising matrix to
powers

0.049966 0.607254 0.247661 0.095119

Step 2:
Multiply times the Eigenvector
(Calc'd from raising matrix to powers)

Multiplied 0.799448 0.93690 1.585032 1.07802 **4.39940694**

λ_{\max}

Step 3:
Add across....)

We then use λ_{\max} to figure out how
inconsistent the matrix is.

Finally, we calculate the Consistency Ratio and from that the Consistency Index (next page).

- The Consistency Ratio
 - $\lambda \text{ max} = 4.39940694$
- The formula for calculating the consistency ratio is:

$$\frac{\lambda \text{ max} - (\text{number of elements in matrix})}{(\text{number of elements in matrix}) - 1}$$

We divide by $n-1$ because that is the inconsistency that was introduced by all other roots other than $\lambda \text{ max}$

$$\frac{4.39940694 - 4}{3}$$

$$\text{Consistency Ratio} = .13313565$$

Consistency Index Calculation

- The Consistency Index is calculated by simply dividing the Consistency Ratio (CR) by the Consistency Ratio of a random matrix of the same size
- The Consistency Ratio of the random matrix is the perfect model of inconsistency (it's random!)

Our Matrix' Consistency Ratio	<u>.13313565</u>
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Random 4 x 4 Matrix' Consistency Ratio	.89
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Consistency Index = .1494382

Why Inconsistency of 10%?

- The notion of order of magnitude is essential.
- When one has a numerical value between 1 and 10
 - A change of a whole integer value (i.e. from 1 to 2) then it is something else. You've changed its identity.
 - If the change is 1% or less it would be too small
 - However, if this perturbation is a decimal (one order of magnitude or smaller) it is acceptable because you are sensitive to it but it doesn't entirely change the magnitude of the number.
 - Changes that are too large can cause dramatic change in our understanding, too small cause no change in our understanding.
 - Therefore we are left with one order of magnitude change = 10%.

Conclusion

- The Eigenvector provides the priorities of the elements in AHP
- The judgments that go into a decision will most often have inconsistency -- you need to know both the priorities and the inconsistency.
- The inconsistency index is a measure compared to “random” inconsistency
- Your level of inconsistency should be $< 10\%$

More questions? Call 703-399-2100 or go to

<http://www.decisionlens.com>